Površinski integral prve vrste

Trebamo izračunati integral $\iint_S f(x,y,z)\,dS$ gdje je S - površ u prostoru.

I način:

Ako je D projekcija površi $S: z=z(x,y)$ na xOy ravan tada

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

II način:

L je projekcija površi $S: y=y(x,z)$ na xOz ravan

$$\iint_S f(x,y,z)\,dS = \iint_L f(x,y(x,z),z)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$$

III način:

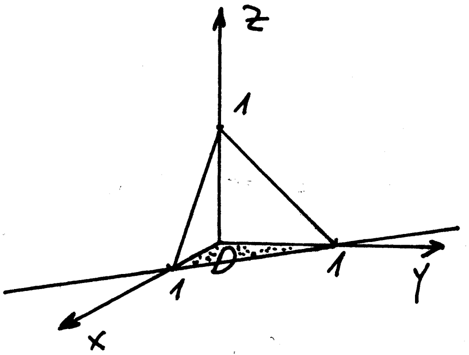
Neka je C projekcija površi $S: x=x(y,z)$ na yOz ravan

$$\iint_S f(x,y,z)\,dS = \iint_C f(x(y,z),y,z)\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

#) Izračunati površinski integral $I=\iint\limits_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

Rj.

$x+y+z=1$ je ravan koja na x, y i z osi odsijeca 1.

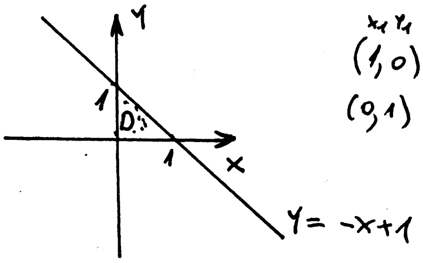


Ako je S data površ opisana jednačinom $z=z(x,y)$ i ako je D projekcija površi S na xOy ravan tada:

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

U našem slučaju $z=1-x-y$, $\frac{\partial z}{\partial x}=-1$, $\frac{\partial z}{\partial y}=-1$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+1+1}=\sqrt{3}$$



$x_1\ y_1$

$(1,0)$

$(0,1)$

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y=-x+1$$

$$D:\begin{cases}0\le x\le 1\\ 0\le y\le -x+1\end{cases}$$

1
1 1
1 2 1
1 3 3 1

Sad imamo

$$I=\iint\limits_S xyz\,dS=\sqrt{3}\iint\limits_D x\cdot y\cdot(1-x-y)\,dx\,dy=\sqrt{3}\int_0^1 x\,dx\int_0^{-x+1}(y-xy-y^2)\,dy=$$

$$=\sqrt{3}\int_0^1 x\left(\frac{1}{2}y^2\Big|_0^{-x+1}-x\cdot\frac{1}{2}y^2\Big|_0^{(-x+1)}-\frac{1}{3}y^3\Big|_0^{(-x+1)}\right)dx=$$

$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x\overbrace{(-x+1)^2}^{x^2-2x+1}-\frac{1}{2}x^2\overbrace{(-x+1)^2}^{x^2-2x+1}-\frac{1}{3}x\overbrace{(-x+1)^3}^{-x^3+3x^2-3x+1}\right)dx=$$

rješenje

$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x^3\cancel{-x^2}+\frac{1}{2}x-\frac{1}{2}x^4\cancel{+x^3}-\frac{1}{2}x^2+\frac{1}{3}x^4\cancel{-x^3}\cancel{+x^2}-\frac{1}{3}x\right)dx$$

$$=\sqrt{3}\int_0^1\left(-\frac{1}{6}x^4+\frac{1}{2}x^3-\frac{1}{2}x^2+\frac{1}{6}x\right)dx=\sqrt{3}\underbrace{\left(-\frac{1}{6}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{2}\right)}_{\frac{1}{120}}=\frac{\sqrt{3}}{120}$$

(#) Izračunati površinski integral $\iint\limits_{(S)} \sqrt{-x^2+4}\, dS$, gdje je (S) omotač površi $\dfrac{x^2}{4}+\dfrac{y^2}{4}=\dfrac{z^2}{9}$, $0 \le z \le 3$.

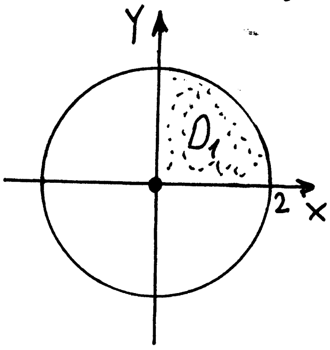
Rj. Skicirajmo površ $\dfrac{x^2}{4}+\dfrac{y^2}{4}=\dfrac{z^2}{9}$, $0 \le z \le 3$

u xOy ravni

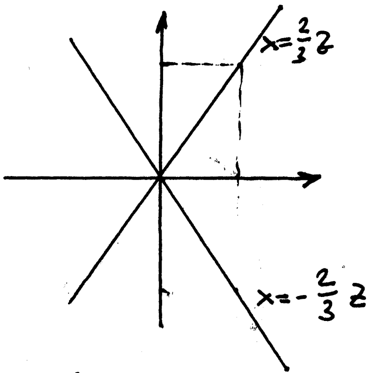
$$\frac{x^2}{4}+\frac{y^2}{4}=0$$

za $z=0$, $x^2+y^2=0$

tačka $(0,0)$



u xOz ravni



$$\frac{x^2}{4}=\frac{z^2}{9}$$

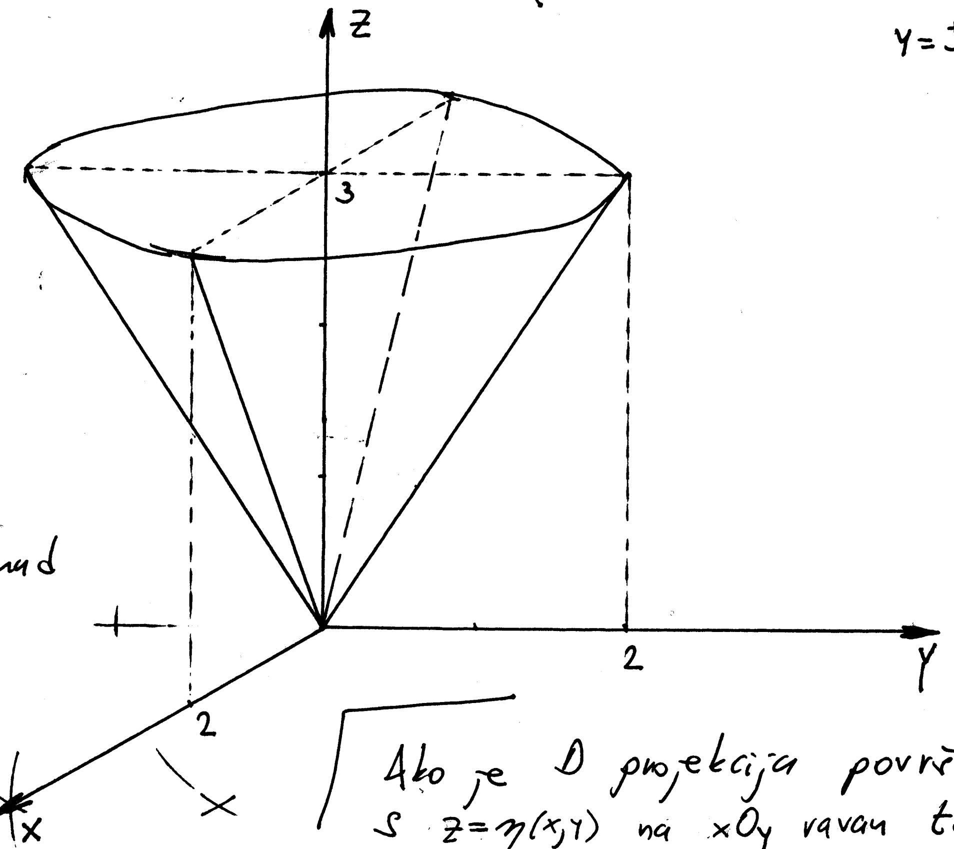
$$x^2=\frac{4}{9}z^2$$

$$x=\pm\frac{2}{3}z$$

yOz ravan

$$y=\pm\frac{2}{3}z$$

Za $z=3$ $\quad x^2+y^2=4$



$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$$

$$z^2=\frac{9}{4}(x^2+y^2)$$

Kako je data površ iznad xOy ravni

$$z=\frac{3}{2}\sqrt{x^2+y^2}$$

$$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{3x}{2\sqrt{x^2+y^2}} \qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_D f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)}=\frac{13x^2+13y^2}{4(x^2+y^2)}=\frac{13}{4}$$

Primjetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = \frac{\sqrt{13}}{2} \iint\limits_{D} \sqrt{-x^2+4}\, dx\,dy = 4 \cdot \frac{\sqrt{13}}{2} \iint\limits_{D_1} \sqrt{4-x^2}\, dx\,dy$$

gde je $D_1 : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = 2\sqrt{13} \int_0^2 \sqrt{4-x^2}\, dx \int_0^{\sqrt{4-x^2}} dy = 2\sqrt{13} \int_0^2 (4-x^2)\, dx =$$

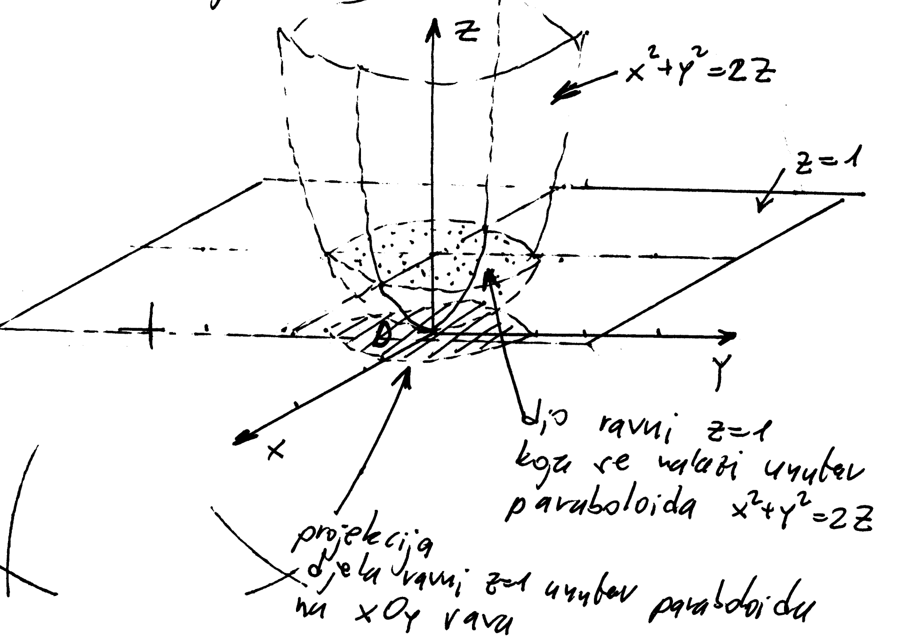
$$= 2\sqrt{13} \left(4x \Big|_0^2 - \frac{1}{3} x^3 \Big|_0^2 \right) = 2\sqrt{13} \left(8 - \frac{8}{3} \right) = 2\sqrt{13} \cdot \frac{\overset{16}{24-8}}{3}$$

$$= \frac{32}{3} \sqrt{13}$$ traženo rješenje

(#) Izračunati površinski integral prvog tipa

$$\iint\limits_{W}(x^2+y^2)\,dS,$$ gdje je W - površina djela ravni $z=1$ koja se nalazi unutar paraboloida $x^2+y^2=2z$.

Rj. Skicirajmo paraboloid $x^2+y^2=2z$ i ravan $z=1$.



Prisjetimo se kako se računa površinski integral prvog tipa

$$\iint\limits_{W} f(x,y,z)\,dS = \iint\limits_{D} f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

gdje je D projekcija površi W na xOy ravan, a W je opisana formulom $z=z(x,y)$.

Projekcija površi W na xOy ravan u našem slučaju je D: unutrašnjost kruga $x^2+y^2=2$.

W: $z=1 \Rightarrow \frac{\partial z}{\partial x}=0$ i $\frac{\partial z}{\partial y}=0$

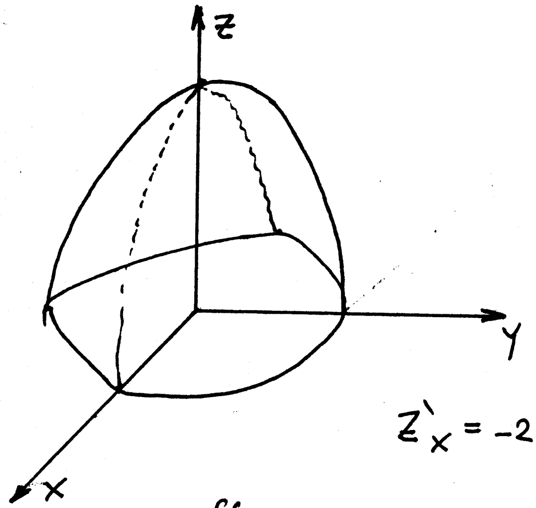
$$\iint\limits_{W}(x^2+y^2)\,dS=\iint\limits_{D}(x^2+y^2)\sqrt{1+0+0}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\\ x^2+y^2=r^2\end{array}\quad D\xrightarrow{\text{transf.}}D':\begin{cases}0\le r\le\sqrt{2}\\ 0\le\varphi\le 2\pi\end{cases}\right|$$

$$=\iint\limits_{D'} r^2\,r\,dr\,d\varphi=\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}r^3\,dr=\varphi\Big|_0^{2\pi}\cdot\frac{1}{4}r^4\Big|_0^{\sqrt{2}}=2\pi\cdot\frac{1}{4}\cdot 4=2\pi$$

(#) Izračunati $\iint_S U(x,y,z)\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy ravni i $U(x,y,z)$ je jednako a) 1

b) x^2+y^2 c) $3z$.

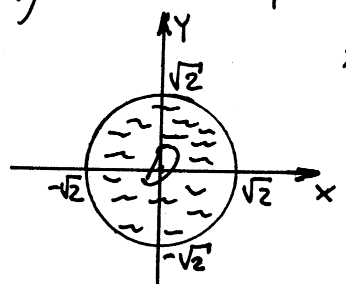
Rj. $\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$ gdje je oblast D projekcija površi S na xOy ravan



$z=2-(x^2+y^2)$

Projekcija na xOy ravan

$x^2+y^2=2$



$z_x' = -2x$

$z_y' = -2y$

$$\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+4x^2+4y^2}\,dx\,dy$$

a) $U(x,y,z)=1$ $\quad I=\iint_D \sqrt{1+4x^2+4y^2}\,dx\,dy$

Da izračunamo ovo transformišimo na polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$

$$D': \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2} \end{cases} \qquad dx\,dy = r\,d\varphi\,dr$$

$$I=\iint_{D'}\sqrt{1+4r^2}\cdot r\,dr\,d\varphi =$$

$$=\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\sqrt{1+4r^2}\cdot r\,dr\right]d\varphi = \left|\begin{matrix} 1+4r^2=t^2 & r=0 \Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{matrix}\right| =$$

$$=\int_0^{2\pi}\left[\int_1^3 t\cdot\frac{1}{4}t\,dt\right]d\varphi = \frac{1}{4}\int_0^{2\pi}\frac{1}{3}t^3\Big|_1^3\,d\varphi = \frac{1}{12}\cdot\varphi\Big|_0^{2\pi}\cdot 26 = \frac{13}{6}\cdot 2\pi = \frac{13\pi}{3}$$

b) vježbu $\quad I=\iint_D (x^2+y^2)\sqrt{1+4x^2+4y^2} = \iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi = \frac{149}{30}$

c) vjež. $\quad I=\frac{111\pi}{10}$

1. Izračunati površinski integral:

a) $I = \iint\limits_{\sigma} (6x + 4y + 3z)ds$, gdje je σ oblast ravni x+2y+3z = 6, u prvom oktantu;

b) $K = \iint\limits_{W} (y + z + \sqrt{a^2 - x^2})ds$, gdje je W površina cilindra $x^2 + y^2 = a^2$, koja se nalazi između ravni z = 0 i z = h.

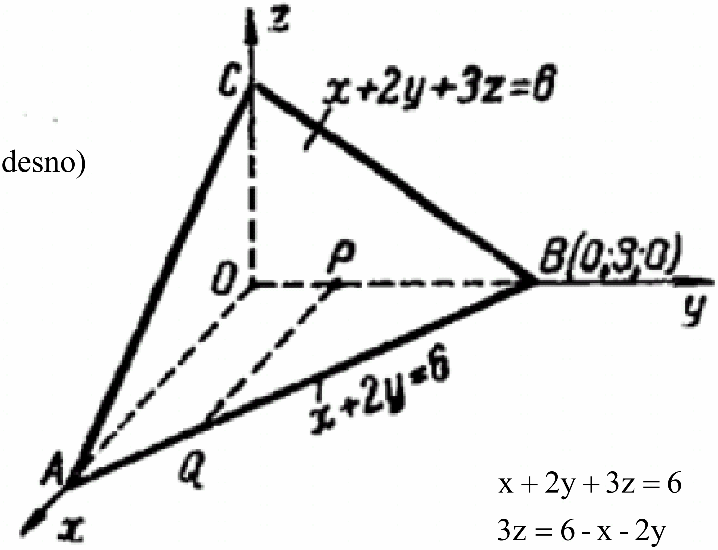
Rješenja:

a) Skicirajmo oblast σ (vidi sliku desno)

x+2y+3z = 6/:6

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

segmentni oblik jednačine ravni



$$x + 2y + 3z = 6$$
$$3z = 6 - x - 2y$$
$$z = \frac{1}{3}(6 - x - 2y)$$

$$\iint\limits_{\sigma} f(x,y,z)ds = \iint\limits_{D} f(x,y,z(x,y)) \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

$$\frac{\partial z}{\partial x} = -\frac{1}{3}$$
$$\frac{\partial z}{\partial y} = -\frac{2}{3}$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{14}{9}} = \frac{\sqrt{14}}{3}$$

Projekcija na xOy ravan izgleda: Nacrtati projekciju (uputa:vidi xOy ravan sa slike iznad).

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$
$$\frac{x-6}{0-6}=\frac{y-0}{3-0}$$
$$\frac{x-6}{-6}=\frac{y}{3}$$
$$3x-18=-6y$$
$$3x=18-6y$$
$$x=6-2y$$

$$D:\begin{cases}0\le y\le 3\\ 0\le x\le 6-2y\end{cases}$$

$$I=\iint\limits_{\sigma}(6x+4y+3z)ds=\frac{\sqrt{14}}{3}\iint\limits_{D}(6x+4y+6-x-2y)dxdy=\frac{\sqrt{14}}{3}\iint\limits_{D}(5x+2y+6)dxdy=$$
$$\frac{\sqrt{14}}{3}\int_0^3 dy\int_0^{6-2y}(5x+2y+6)dx=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}x^2\Big|_0^{6-2y}+2xy\Big|_0^{6-2y}+6x\Big|_0^{6-2y}\right)dy=$$
$$=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(6-2y)^2+2\cdot(6-2y)\cdot y+6\cdot(6-2y)\right)dy=$$
$$=\frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(36-24y+4y^2)+12y-4y^2+36-12y\right)dy=$$
$$=\frac{\sqrt{14}}{3}\int_0^3(6y^2-60y+126)dy=2\sqrt{14}\int_0^3(y^2-10y+21)dy=$$
$$=2\sqrt{14}\cdot\left(\frac{y^3}{3}\Big|_0^3-10\frac{y^2}{2}\Big|_0^3+21y\Big|_0^3\right)=2\sqrt{14}\cdot(9-45+63)=54\sqrt{14}$$

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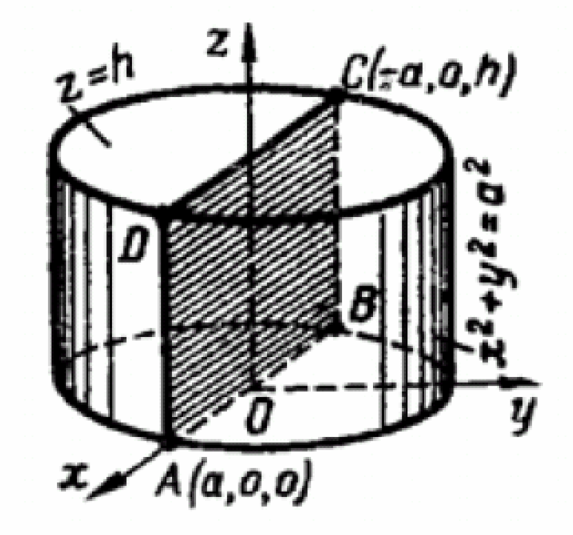
b) $K=\iint\limits_{W}(y+z+\sqrt{a^2-x^2})ds$ $\quad x^2+y^2=a^2 \quad$ z = 0 i z = h

Skicirajmo oblast W (vidi sliku na sljedećoj stranici)

$$\iint\limits_{W}f(x,y,z)ds=\iint\limits_{D}f(x,y(x,z),z)\cdot\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}dxdy$$



$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$

$$|y|=\sqrt{a^2-x^2}$$
$$y=\sqrt{a^2-x^2}$$
i
$$y=-\sqrt{a^2-x^2}$$

$$K=K_1+K_2$$
$$\frac{\partial y}{\partial x}=-\frac{x}{\sqrt{a^2-x^2}}$$
$$\frac{\partial y}{\partial z}=0$$

$$D:\begin{cases}-a\le x\le a\\ 0\le z\le h\end{cases}$$

$$ds=\sqrt{1+\left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,dxdz=\sqrt{1+\frac{x^2}{a^2-x^2}}\,dxdz=\frac{adxdz}{\sqrt{a^2-x^2}}$$

$$K_1=\iint\limits_W\left(y+z+\sqrt{a^2-x^2}\right)ds=\iint\limits_D\left(\sqrt{a^2-x^2}+z+\sqrt{a^2-x^2}\right)\frac{adxdz}{\sqrt{a^2-x^2}}=$$
$$=a\iint\limits_D\left(2+\frac{z}{\sqrt{a^2-x^2}}\right)dxdz=2a\int\limits_{-a}^{a}dx\int\limits_0^h dz+a\int\limits_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\int\limits_0^h zdz=$$
$$=2a\cdot 2a\cdot h+\frac{ah^2}{2}\int\limits_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=a\Rightarrow t=\dfrac{\pi}{2}\\ y=-a\Rightarrow t=-\dfrac{\pi}{2}\end{array}\right|=4a^2h+\frac{ah^2}{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{\sqrt{a^2-a^2\sin^2 t}}=$$
$$=4a^2h+\frac{ah^2}{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{a\sqrt{1-\sin^2 t}}=4a^2h+\frac{ah^2}{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos tdt}{\cos t}=4a^2h+\frac{ah^2}{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt=$$
$$=4a^2h+\frac{a^2h}{2}\cdot t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=4a^2h+\frac{ah^2\pi}{2}$$

$$y = -\sqrt{a^2 - x^2}$$

$$\frac{\partial y}{\partial x} = \frac{x}{\sqrt{a^2 - x^2}}$$

$$\frac{\partial y}{\partial z} = 0$$

$$ds = \frac{adxdy}{\sqrt{a^2 - x^2}}$$

$$K_2 = \iint\limits_W \left(y + z + \sqrt{a^2 - x^2}\right) ds = \iint\limits_D \left(-\sqrt{a^2 - x^2} + z + \sqrt{a^2 - x^2}\right) \frac{adxdz}{\sqrt{a^2 - x^2}} =$$

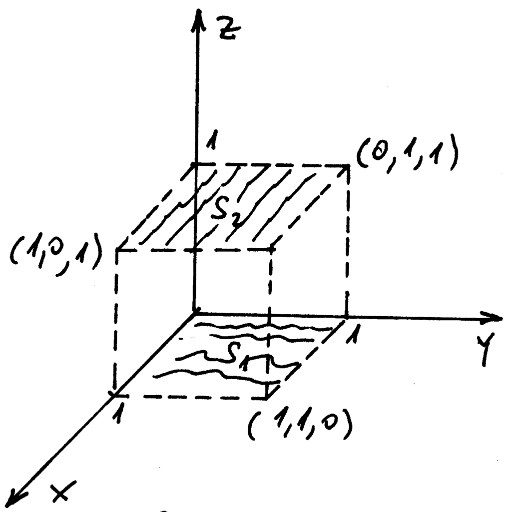
$$= \iint\limits_D z \frac{adxdz}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}} \int_0^h zdz = a \int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}} \cdot \frac{z^2}{2} \Bigg|_0^h = \frac{ah^2}{2} \int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}} =$$

$$= \left| \begin{array}{l} x = a \sin t \\ dx = a \cos t dt \\ x = a \Rightarrow t = \frac{\pi}{2} \\ x = -a \Rightarrow t = -\frac{\pi}{2} \end{array} \right| = \frac{ah^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a \cos t dt}{\sqrt{a^2 - a^2 \sin^2 t}} = \frac{ah^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \frac{ah^2}{2} \cdot t \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{ah^2 \pi}{2}$$

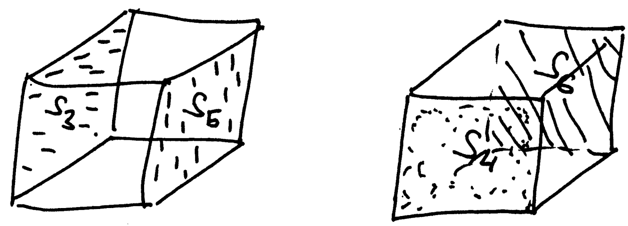
$$K = 4a^2 h + \frac{ah^2 \pi}{2} + \frac{ah^2 \pi}{2} = 4a^2 h + ah^2 \pi = ah(4a + \pi h)$$

(#) Izračunati površinski integral $\iint\limits_S (x+y+z)\,dS$ gdje je S površina kocke $0\le x\le 1$, $0\le y\le 1$ i $0\le z\le 1$.

Rj.



Ako strane kocke označim sa S_1, S_2, S_3, S_4, S_5 i S_6



imamo:

$$\iint\limits_S (x+y+z)\,dS = \iint\limits_{S_1} (x+y+z)\,dx\,dy + \iint\limits_{S_2} (x+y+z)\,dx\,dy + \iint\limits_{S_3} (x+y+z)\,dx\,dz$$

$$+ \iint\limits_{S_4} (x+y+z)\,dy\,dz + \iint\limits_{S_5} (x+y+z)\,dx\,dz + \iint\limits_{S_6} (x+y+z)\,dy\,dz$$

$$\iint\limits_{S_1} (x+y+z)\,dx\,dy + \iint\limits_{S_2} (x+y+z)\,dx\,dy = \int_0^1 \left[\int_0^1 (x+y)\,dy\right] dx + \int_0^1 \left[\int_0^1 (x+y+1)\,dy\right] dx =$$

$$\int_0^1 \left(xy\Big|_0^1 + \frac{1}{2}y^2\Big|_0^1\right) dx + \int_0^1 \left(xy\Big|_0^1 + \frac{1}{2}y^2\Big|_0^1 + y\Big|_0^1\right) dx = \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}x\Big|_0^1 + \frac{1}{2}x\Big|_0^1$$

$$+ \frac{1}{2}x\Big|_0^1 + x\Big|_0^1 = 3$$

Prema tome: $\iint\limits_S (x+y+z)\,dS = 9$.

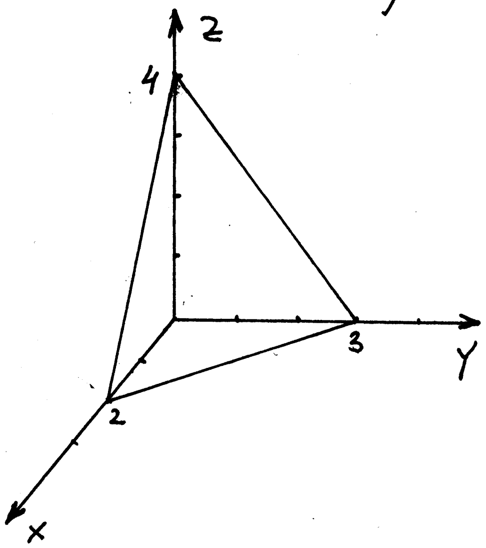
(#) Izračunati površinski integral $\iint (z+2x+\frac{4}{3}y)\,dS$ gdje je S dio ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ u prvom oktantu.

Rj.

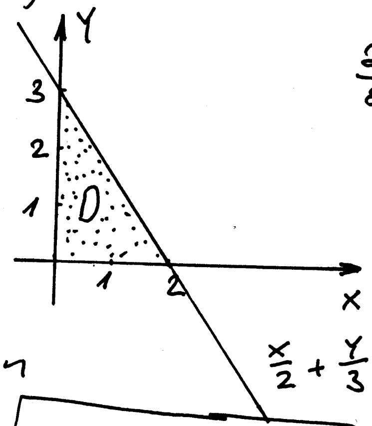
$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ segmentni oblik jednačine ravni

$\frac{z}{4}=1-\frac{x}{2}-\frac{y}{3} \quad /\cdot 4$

$z=4-2x-\frac{4}{3}y$



Projekcija na xOy ravan izgleda



$\frac{\partial z}{\partial x}=-2$

$\frac{\partial z}{\partial y}=-\frac{4}{3}$

S' projekcija površi S na xOy ravan

$\frac{x}{2}+\frac{y}{3}=1$

$$I=\iint_S f(x,y,z)\,dS=\iint_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

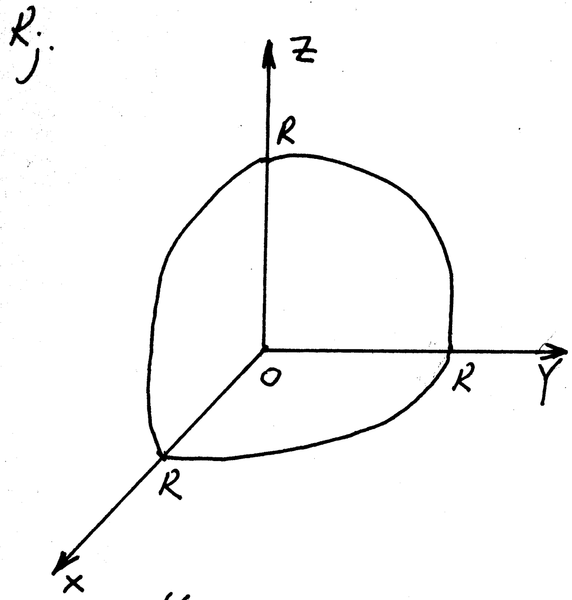
$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+4+\frac{16}{9}}=\sqrt{\frac{61}{9}}=\frac{\sqrt{61}}{9}$$

$$\iint_S (z+2x+\tfrac{4}{3}y)\,dS=\iint_D (4-2x-\tfrac{4}{3}y+2x+\tfrac{4}{3}y)\cdot\frac{\sqrt{61}}{3}\,dx\,dy=\frac{4\sqrt{61}}{3}\underbrace{\iint_D dx\,dy}_{\text{površina oblasti } D}$$

$$=\frac{4\sqrt{61}}{3}\cdot\frac{2\cdot 3}{2}=4\sqrt{61}.$$

Izračunati integral $I=\iint\limits_S x\,dS$ gdje je S dio sfere $x^2+y^2+z^2=R^2$ u prvom oktantu.

R.j.



$$x^2+y^2+z^2=R^2$$

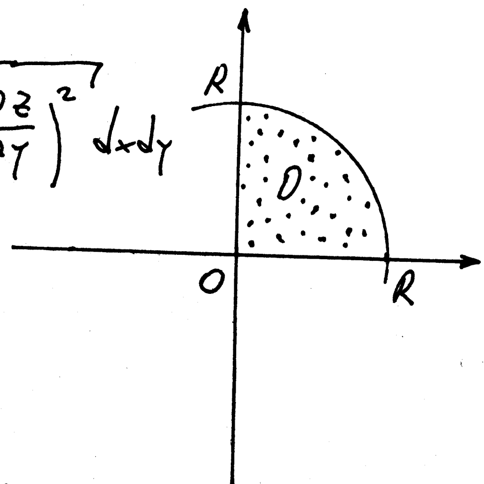
$$z^2=R^2-x^2-y^2$$

$$z=\pm\sqrt{R^2-x^2-y^2}$$

$$z\geq 0 \qquad z=\sqrt{R^2-x^2-y^2}$$

Projekcija površi na xOy ravan

$$\iint\limits_S f(x,y,z)\,dS=\iint\limits_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$



S' projekcija površi S na xOy ravan

$$\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2-y^2}}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}} \qquad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}$$

$$I=\iint\limits_S x\,dS=\iint\limits_D x\cdot\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$ u našem slučaju

$$D':\begin{cases} x=r\cos\varphi\\ y=r\sin\varphi\\ 0\leq\varphi\leq\frac{\pi}{2}\\ 0\leq r\leq R\end{cases} \qquad dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2$$

$$\iint\limits_D \frac{xR}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint\limits_{D'}\frac{r\cos\varphi R\cdot r}{\sqrt{R^2-r^2}}\,dr\,d\varphi$$

$$=R\int_0^{\pi/2}\cos\varphi\left[\int_0^R\frac{r^2}{\sqrt{R^2-r^2}}\,dr\right]d\varphi=\left|\begin{array}{l} r=R\sin t\\ r=0\Rightarrow t=0\\ r=R\Rightarrow t=\frac{\pi}{2}\\ dr=R\cos t\,dt\end{array}\right|=R\int_0^{\pi/2}\cos\varphi\left[\int_0^{\frac{\pi}{2}}\frac{1}{R}\,\frac{R^2\sin^2 t}{\sqrt{1-\sin^2 t}}\,R\cos t\,dt\right]d\varphi$$

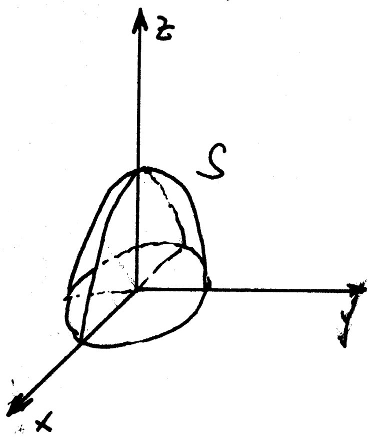
$$=R^3\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\sin^2 t\,dt\right]d\varphi=R^3\int_0^{\pi/2}\cos\varphi\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos 2t)\,dt\right]=R^3\sin\varphi\Big|_0^{\pi/2}\cdot\frac{1}{2}\left(t\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\sin 2t\Big|_0^{\pi/2}\right)=\frac{R^3\pi}{4}$$

\# Izračunati površinski integral $\iint\limits_S 3z\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy-ravni.

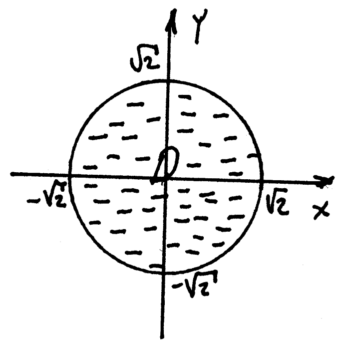
Rj Neka je D projekcija površi S na xOy ravan. Tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Pronađimo projekciju paraboloida $z=2-(x^2+y^2)$ na xOy ravan.



$z=0 \Rightarrow x^2+y^2=2$ krug sa centrom u tački $(0,0)$ poluprečnika $\sqrt{2}$



$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -2y$$

$$I=\iint\limits_S 3z\,dS = 3\iint\limits_D \left[2-(x^2+y^2)\right]\sqrt{1+4x^2+4y^2}\,dx\,dy$$

Da bi smo riješili ovaj dvostruki integral potrebno je uvesti smjenu promjenjivih.

Uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$

$$D': \begin{cases} 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2} \\ dx\,dy = r\,dr\,d\varphi \end{cases}$$ ove granice čitamo sa slike

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$1+4x^2+4y^2=1+4(x^2+y^2)=1+4r^2$

$$I = 3\iint\limits_{D'}(2-r^2)\sqrt{1+4r^2}\cdot r\,dr\,d\varphi = 3\iint\limits_{D'}2r\sqrt{1+4r^2}\,dr\,d\varphi-3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi$$

$$6\iint\limits_{D'}r\sqrt{1+4r^2}\,dr\,d\varphi = 6\int_0^{2\pi}\left[\int_0^{\sqrt{2}}r\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r=0\Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{array}\right| =$$

$$=6\int_0^{2\pi}\left[\int_1^3\frac{1}{4}t^2\,dt\right]d\varphi = 6\cdot\frac{1}{4}\,\varphi\Big|_0^{2\pi}\cdot\frac{t^3}{3}\Big|_1^3 = \frac{3}{2}\cdot\frac{1}{3}\cdot 2\pi\cdot 26 = 26\pi$$

$$3\iint\limits_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi = 3\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\underbrace{r^3}_{r^2\cdot r}\sqrt{1+4r^2}\,dr\right]d\varphi = \left|\begin{array}{ll} 1+4r^2=t^2 & r\,dr=\frac{1}{4}t\,dt \\ 4r^2=t^2-1 & r=0\Rightarrow t=1 \\ r^2=\frac{1}{4}(t^2-1) & r=\sqrt{2}\Rightarrow t=3 \\ 8r\,dr=2t\,dt & \end{array}\right| = \ldots = \frac{111\pi}{10}$$

Zadaci za vježbu

U zadacima 3876—3884 izračunati date integrale.

3876. $\iint\limits_S \left(z+2x+\frac{4}{3}y\right) dq$, pri čemu je S deo ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ koji se nalazi u prvom oktantu.

3877. $\iint\limits_S xyz\, dq$, pri čemu je S deo ravni $x+y+z=1$ koja leži u prvom oktantu.

3878. $\iint\limits_S x\, dq$, pri čemu je S deo sfere $x^2+y^2+z^2=R^2$ koji se nalazi u prvom oktantu.

3879. $\iint\limits_S y\, dq$, po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3880. $\iint\limits_S \sqrt{R^2-x^2-y^2}\, dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3881. $\iint\limits_S x^2 y^2\, dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3882. $\iint\limits_S \frac{dq}{r^2}$, pri čemu je S deo cilindra $x^2+y^2=R^2$ ograničen ravnima $z=0$ i $z=H$, a r je odstojanje tačke na površini od koordinatnog početka.

3883. $\iint\limits_S \frac{dq}{r^n}$ po sferi $x^2+y^2+z^2=R^2$, pri čemu je r odstojanje tačke na sferi od nepomične tačke $P(0,\ 0,\ c)$, $(c>R)$.

3884. $\iint\limits_S \frac{dq}{r}$, pri čemu je S deo hiperboličnog paraboloida $z=xy$, isečen cilindrom $x^2+y^2=R^2$, a r je odstojanje tačke na površi S od z-ose.

3885*. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka odstojanju te tačke od nekog određenog prečnika sfere.

3886. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka kvadratu odstojanja te tačke od nekog određenog prečnika sfere.

Rješenja

3876. $4\sqrt{61}$. **3877.** $\frac{\sqrt{3}}{120}$. **3878.** $\frac{\pi R^3}{4}$.

3979. 0. **3880.** πR^3. **3881.** $\frac{2\pi R^6}{15}$.

3882. $2\pi \operatorname{arctg}\frac{H}{R}$. **3883.** $\frac{2\pi R}{c(n-2)}\left[\frac{1}{(c-R)^{n-2}}-\frac{1}{(c+R)^{n-2}}\right]$ za $n\neq 2$; $\frac{2\pi R}{c}\ln\frac{c+R}{c-R}$ za $n=2$.

3884. $\pi\left[R\sqrt{R^2+1}+\ln\left(R+\sqrt{R^2+1}\right)\right]$.

3885*. $\pi^2 R^3$. Primeniti sferne koordinate. **3886.** $\frac{8}{3}\pi R^4$.

Površinski integrali II vrste

Obično su oblika: $\iint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy$

Uvijek ga svodimo na dvostruki integral.
S je neka data površina. Početni integral se obično podjeli na tri dijela $\iint\limits_S P(x,y,z)\,dy\,dz$, $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\iint\limits_S R(x,y,z)\,dx\,dy$.

$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ je vektor normale na površinu S gdje su α, β i γ uglovi koje zaklapa vektor normale sa x, y i z osom.
Kad računamo $\iint\limits_S P(x,y,z)\,dy\,dz$ treba uzeti u obzir predznak broja $\cos\alpha$. Ako je $\cos\alpha < 0$ ispred integrala stavljamo − (minus), ako je $\cos\alpha > 0$ ispred integrala stavljamo + (plus) i ako je $\cos\alpha = 0$ tada $\iint\limits_S P(x,y,z)\,dy\,dz = 0$.
Analogno uzimamo vrijednost $\cos\beta$ za $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\cos\gamma$ za $\iint\limits_S R(x,y,z)\,dx\,dy$. $I = I_1 + I_2 + I_3$
Integral I_1 rješavamo projekcijom površi S na yOz ravan, integral I_2 projekcijom na xOz ravan i integral I_3 $I_3 = \iint\limits_S R(x,y,z)\,dx\,dy$ projekcijom površi S na xOz ravan.

Kod površinskih integrala II vrste mora se označiti koju stranu površi uzimamo. Zavisi od toga sa koje strane vektor normale djeluje (ili sa unutrašnje ili sa spoljašnje oblasti površi).
Kod izbora površi S po kojoj se integrira mora se precizirati da li se uzima vanjska ili unutrašnja strana površi, jer prelaskom na suprotnu stranu integral mijenja predznak.

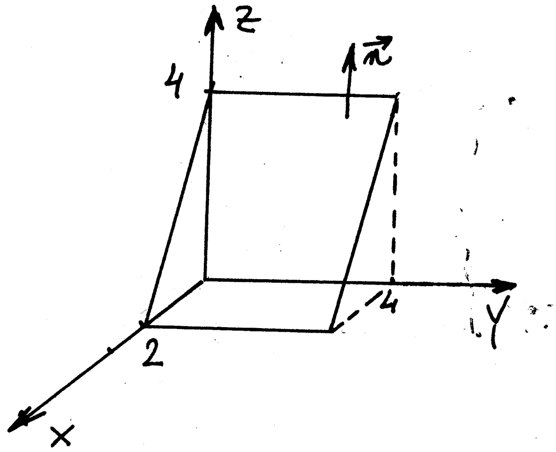
(#) Izračunati $\iint\limits_S z\,dx\,dy + x\,dz\,dx + y\,dy\,dz$ pri čemu je S gornja strana ravni $2x+z=4$, $0<y<4$ u prvom oktantu.

Rj.

$2x+z=4 \quad /:4$

$\frac{x}{2}+\frac{z}{4}=1$ segmentni oblik jednačine ravni

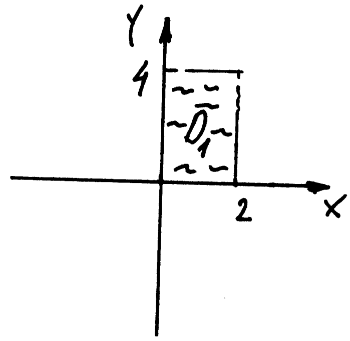
$z=4-2x$



$\vec{n}=(2,0,1)$ vektor normale ravni

$|\vec{n}|=\sqrt{5}$ $\quad \vec{n}_0=\frac{\vec{n}}{|\vec{n}|}=\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$

$\cos\alpha \quad \cos\beta \quad \cos\gamma$



$I_1=\iint\limits_S z\,dx\,dy$ projeciramo površ na xOy ravan $\quad D_1: \begin{cases} 0\le x\le 2 \\ 0\le y\le 4 \end{cases}$

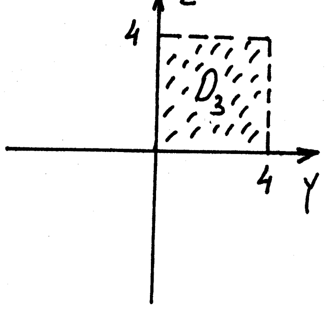
kako je $\cos\gamma>0 \Rightarrow I_1=+\iint\limits_{D_1}(4-2x)\,dx\,dy=$

$=\int\limits_0^4\left[\int\limits_0^2(4-2x)\,dx\right]dy=\int\limits_0^4\left(4x\Big|_0^2-2\cdot\frac{1}{2}x^2\Big|_0^2\right)dy=\int\limits_0^4(8-4)\,dy=4y\Big|_0^4=16$

$I_2=\iint\limits_S x\,dz\,dx$ (gledamo ugao β)

Kako je $\cos\beta=0 \Rightarrow I_2=0$

$I_3=\iint\limits_S y\,dy\,dz$ (gledamo ugao α) $\cos\alpha>0 \Rightarrow I_3=+\iint\limits_{D_3} y\,dy\,dz$



$D_3: \begin{cases} 0\le y\le 4 \\ 0\le z\le 4 \end{cases}$

$I_3=\int\limits_0^4\left[\int\limits_0^4 y\,dy\right]dz=\int\limits_0^4\frac{1}{2}y^2\Big|_0^4\,dz$

$=\frac{1}{2}\cdot 16\cdot z\Big|_0^4=32$

$\iint\limits_S z\,dx\,dy+x\,dz\,dx+y\,dy\,dz=16+0+32=48$

(#) Izračunati površinski integral druge vrste

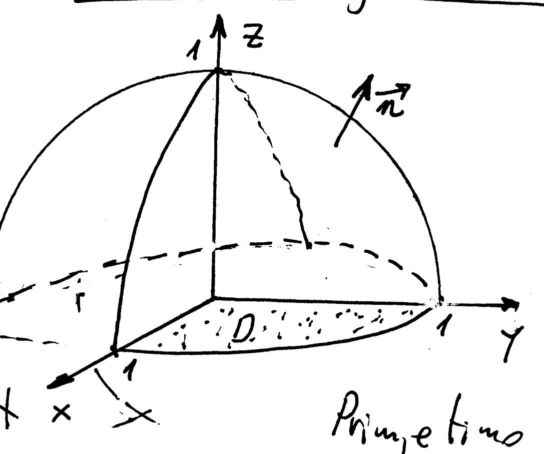
$$I = \iint_S xyz\, dx\, dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x \geq 0$, $y \geq 0$.

Rj. Prisjetimo se: Neka je površ S data u obliku $z = \eta(x,y)$. Tada

$$\iint_S R(x,y,z)\, dx\, dy = \pm \iint_D R(x,y,\eta(x,y))\, dx\, dy \quad \text{gdje}$$

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$,
  - $\cos\gamma > 0 \Rightarrow +$
  - $\cos\gamma < 0 \Rightarrow -$
  - $\cos\gamma = 0 \Rightarrow 0$
- D je ortogonalna projekcija površi S na xOy ravan



$$z^2 = 1 - x^2 - y^2$$

$$z = \pm\sqrt{1-x^2-y^2}$$

kako je $x \geq 0$, $y \geq 0$ to je $z = \sqrt{1-x^2-y^2}$

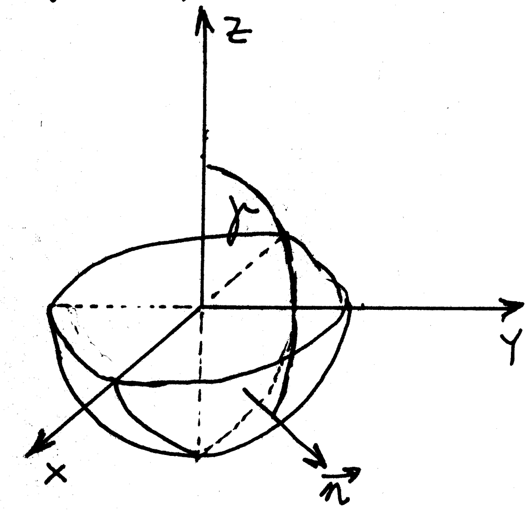
Primjetimo da je $0 < \gamma < 90^\circ \Rightarrow \cos\gamma > 0$

$$\iint_S xyz\, dx\, dy = \iint_D xy\sqrt{1-x^2-y^2}\, dx\, dy = \left| \begin{array}{ll} \text{uvedimo polarne koordinate} & \\ x = r\cos\varphi & 1-x^2-y^2 = 1-r^2 \\ y = r\sin\varphi & \\ dx\,dy = r\,dr\,d\varphi & D \xrightarrow{\text{transf.}} D' : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases} \end{array} \right|$$

$$= \iint_{D'} r^3 \sin\varphi\cos\varphi\sqrt{1-r^2}\, dr\, d\varphi = \int_0^1 r^3\sqrt{1-r^2}\, dr \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi = \underset{\dots}{\overset{\text{za vježbu}}{}} = \frac{2}{15}\cdot\frac{1}{2} = \frac{1}{15}$$

Izračunati $\iint\limits_S x^2y^2z\,dx\,dy$ gdje je S-vanjska strana donje polovine sfere $x^2+y^2+z^2=R^2$.

Rj.



Kako imamo $dx\,dy$: zanima nas ugao γ (γ je ugao koji vektor normale $\vec{n}$ na površ zaklapa sa z-osom).

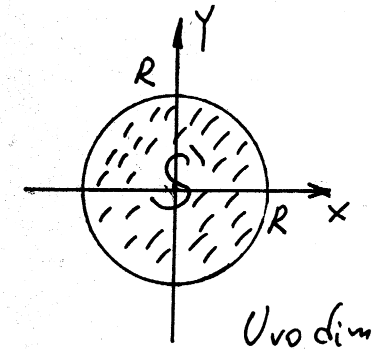
$$\gamma > \frac{\pi}{2} \Rightarrow \cos\gamma < 0$$

$$z^2 = R^2 - x^2 - y^2$$

$$z<0 \qquad z = -\sqrt{R^2-x^2-y^2}$$

Da smo imali čitavu sferu tada bi integral podijelili na dva dijela za gornji i za donji dio sfere.

Gledamo projekciju površi S na xOy ravan:



$$S': \ x^2+y^2 \le R^2$$

$$\iint\limits_S x^2y^2z\,dx\,dy = -\iint\limits_{S'} x^2y^2(-\sqrt{R^2-x^2-y^2}\,dx\,dy$$

Uvodimo polarne koordinate

$x = r\cos\varphi$
$y = r\sin\varphi$

$$D: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le R \\ dx\,dy = r\,d\varphi\,dr \end{cases} \qquad x^2+y^2=r^2$$

$$\iint\limits_S x^2y^2z\,dx\,dy = \iint\limits_{S'} x^2y^2\sqrt{R-x^2-y^2}\,dx\,dy = \iint r^2\cos^2\varphi\, r^2\sin^2\varphi\sqrt{R^2-r^2}\cdot r\,d\varphi\,dr$$

$$= \int_0^{2\pi}\cos^2\varphi\sin^2\varphi\left[\int_0^R r^5\sqrt{R^2-r^2}\,dr\right]d\varphi \overset{(*)}{=} \frac{8R^7}{105}\cdot\frac{\pi}{4} = \frac{2\pi R^7}{105}$$

(*)

$$\int_0^R r^5\sqrt{R^2-r^2}\,dr = \int_0^R r^4\sqrt{R^2-r^2}\,r\,dr = \left|\begin{matrix} R^2-r^2=t^2 & r=0 \Rightarrow t=R \\ -2r\,dr = 2t\,dt & r=R \Rightarrow t=0 \\ r\,dr = -t\,dt & \end{matrix}\right| = \int_0^R (R^2-t^2)^2\cdot\sqrt{t^2}\cdot t\,dt = \dots = \frac{8R^7}{105}$$

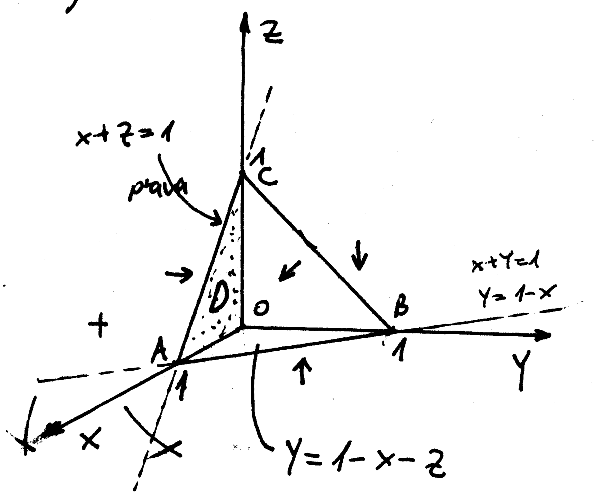
$$\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\,d\varphi = \int_0^{2\pi}\frac14(2\cos\varphi\sin\varphi)^2\,d\varphi = \frac14\int_0^{2\pi}\sin^2 2\varphi\,d\varphi = \frac14\int_0^{2\pi}\frac12(1-\cos4\varphi)\,d\varphi = \frac18\left(\varphi\Big|_0^{2\pi} - \frac14\sin4\varphi\Big|_0^{2\pi}\right) = \frac{\pi}{4}$$

(7) Izračunati površinski integral $K=\oiint\limits_{-W} y\,dx\,dz$ gdje je

W - površina tetraedra ograničenog ravnima $x+y+z=1$, $x=0$, $y=0$ i $z=0$.

Rj: integral oblika $\iint\limits_{W} R(x,y,z)\,dx\,dz$ zovemo površinski integral drugog tipa. Računamo ga tako što napravimo projekciju D površi W na xOz ravan i odredimo predznak broja $\cos\beta$ gdje je β ugao koji zaklapa vektor normale $\vec{n}$ površi W sa Y-osom.

Skicirajmo naš tetraedar



Kako je u zadatku data oblast $-W$ to posmatramo vektore normale koje odgovaraju unutrašnjim površinama tetraedra

$$K=\oiint\limits_{-W} y\,dx\,dz=\iint\limits_{-\Delta AOC} y\,dx\,dz+\iint\limits_{-\Delta AOB} y\,dx\,dz+\iint\limits_{-\Delta BOC} y\,dx\,dz+\iint\limits_{-\Delta ABC} y\,dx\,dz$$

$$\iint\limits_{-\Delta AOC} y\,dx\,dz=+\iint\limits_{D} 0\cdot dx\,dz=0$$

$$\iint\limits_{-\Delta AOB} y\,dx\,dz=\left|\begin{array}{l}\text{vektor normale } \Delta AOB\\ \text{je okomit na } y\text{-osu}\end{array}\right|=0$$

$$\iint\limits_{-\Delta OBC} y\,dx\,dz=\left|\begin{array}{l}\text{vektor normale } \Delta OBC\\ \text{je okomit na } y\text{-osu}\end{array}\right|=0$$

$$\iint\limits_{-\Delta ABC} y\,dx\,dz = \left| \begin{array}{l} \text{vektor normale } \vec{n}^0 \text{ na} \\ \Delta ABC \text{ sa } Y\text{-osom zaklapa} \\ \text{ugao } \beta \text{ koji je između } 90^\circ \text{ i } 180^\circ \\ \text{ZAŠTO? (vidi sliku)} \\ \cos\beta < 0 \end{array} \right| = -\iint\limits_{D} (1-x-z)\,dx\,dz =$$

$$= -\int_0^1 dx \int_0^{1-x} (1-x-z)\,dz = -\int_0^1 \left(z\Big|_0^{1-x} - xz\Big|_0^{1-x} - \frac{1}{2}z^2\Big|_0^{1-x} \right) dx =$$

$$= -\int_0^1 \Big(1-x - x(1-x) - \frac{1}{2}\underbrace{(1-x)^2}_{1-2x+x^2} \Big)\,dx = -\int_0^1 \left(1 - x - \cancel{x} + x^2 - \frac{1}{2} + \cancel{x} - \frac{1}{2}x^2\right) dx$$

$$= -\int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right) dx = -\left(\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}x\Big|_0^1\right) = -\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) = -\frac{1}{6}$$

traženo rješenje

II način

Možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint\limits_{S} P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy = \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

Ω - oblast koju ograničava površ S

U našem slučaju $P(x,y,z) = R(x,y,z) = 0$

$$Q(x,y,z) = y \quad \Rightarrow \quad \frac{\partial Q}{\partial y} = 1$$

$$K = \oiint\limits_{-W} y\,dx\,dz = -\iiint\limits_{\Omega} dx\,dy\,dz = -\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = -\int_0^1 dx \int_0^{1-x} (1-x-y)\,dy$$

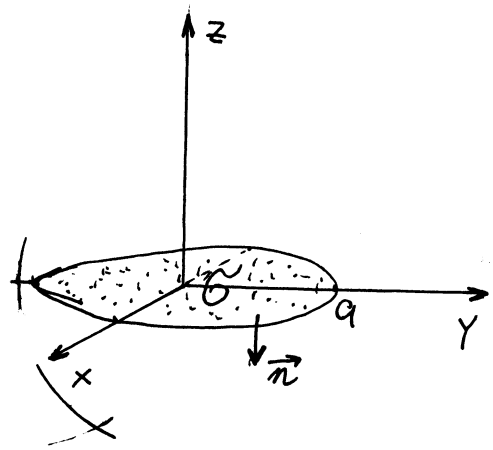
$$= \left| \begin{array}{l} \text{primjetimo da smo sličan} \\ \text{integral već imali u prethodnom} \\ \text{slučaju} \end{array} \right| = \ldots = -\frac{1}{6}$$

traženo rješenje

(#) Izračunati površinski integral drugog tipa (po koordinatama) $I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy$ gdje je

σ - donja strana kruga $x^2+y^2\le a^2$.

Rj: Skicirajmo datu površinu



Prisjetimo se, kako se računa površinski integral drugog tipa, npr. $\iint\limits_{S}R(x,y,z)\,dx\,dy$

posmatramo vektor normale $\vec{n}$ na površ S

ako je $\cos\gamma<0$ gdje je γ ugao između $\vec{n}$ i z-ose naš integral postaje $\iint\limits_{S}R(x,y,z)\,dx\,dy=-\iint\limits_{D}R(x,y,z(x,y))\,dx\,dy$

gdje je D ortogonalna projekcija površi S a $z=z(x,y)$ jednačina površi S

U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ.

Ugao γ je $\gamma=\pi$ (donja strana kruga) tj. $\cos\pi<0$.

$$I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy=-\iint\limits_{D}\sqrt[4]{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvodimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi \qquad D\xrightarrow{\text{transf.}}D':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}\\ dx\,dy=r\,dr\,d\varphi\end{array}\right|$$

$$=-\iint\limits_{D'}\sqrt[4]{r^2}\,r\,dr\,d\varphi=-\int_0^{2\pi}d\varphi\int_0^a r^{\frac{3}{2}}dr=-\int_0^{2\pi}\frac{2}{5}r^{\frac{5}{2}}\Big|_0^a\,d\varphi=-\frac{2}{5}a^{\frac{5}{2}}\varphi\Big|_0^{2\pi}$$

$I=-\frac{4}{5}\pi\sqrt{a^5}$ traženo rješenje

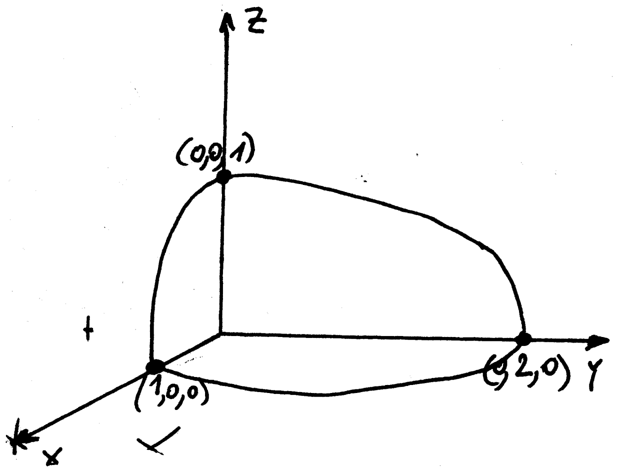
(#) Izračunati površinski integral

$$J=\iint_T z\,dx\,dy+y\,dx\,dz-x^2z\,dy\,dz$$ gdje je T vanjska strana elipsoida $4x^2+y^2+4z^2=4$ koji se nalazi u prvom oktantu.

Rj. skicirajmo elipsoid

$4x^2+y^2+4z^2=4 \quad /:4$

$$\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{1}=1$$



$$J=\iint_T z\,dx\,dy+y\,dx\,dz-x^2z\,dy\,dz$$

Ovo je površinski integral druge vrste. Prisjetimo se kako se računa npr. $\iint_T P(x,y,z)\,dy\,dz$. Neka je $\vec{n}$ vektor normale površi T koji sa x, y i z osom redom zaklapa uglove α, β i γ, i neka je D ortogonalna projekcija površi T na yOz ravan. Tada

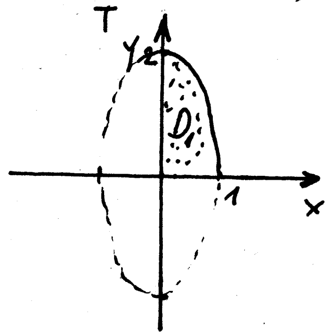
$$\iint_T P(x,y,z)\,dy\,dz=\pm\iint_D P(\eta(y,z),y,z)\,dy\,dz$$ gdje je $+$ ako je $\cos\alpha>0$, $-$ (minus) ako je $\cos\alpha<0$, a $x=\eta(y,z)$ je jednačina koja opisuje površ T.

$$J=\iint_T z\,dx\,dy+y\,dx\,dz-x^2z\,dy\,dz=\iint_T z\,dx\,dy+\iint_T y\,dx\,dz-\iint_T x^2z\,dy\,dz=$$

$$=J_1+J_2-J_3$$

Izračunajmo redom J_1, J_2 i J_3.

$J_1=\iint_T z\,dx\,dy$; vektor normale $\vec{n}$ na T sa z osom zaklapa ugao $\gamma\in(0,\frac{\pi}{2})$ tj. $\cos\gamma>0$



$z=0: \quad 4x^2+y^2=4$

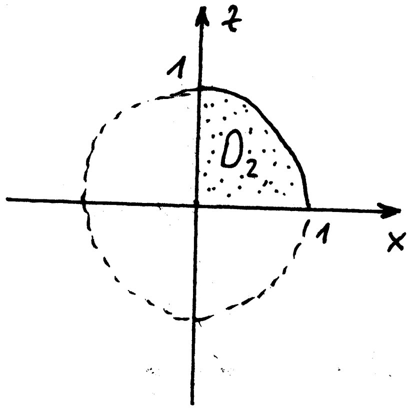
$$D_1: \begin{cases} 0\le x\le 1 \\ 0\le y\le 2\sqrt{1-x^2} \end{cases}$$

D_1 je četvrtina elipse

$P_{elipse}=ab\pi$, $\quad J_1=+2\iint_{D_1} dx\,dy=2\cdot\frac{1}{4}P_{elipse}=\frac{1}{2}\cdot 2\pi=\pi$

$J_2=\iint\limits_T y\,dx\,dz$, vektor normale $\vec{n}$ na površ T sa y-osom zaklapa uglove od 0 do $\pi/2$ (I oktant) pa je $\cos\gamma>0$.

Neka je D_2 ortogonalna projekcija površi T na xOz ravan.



$D_2:\ 4x^2+4z^2=4$

$4x^2+y^2+4z^2=4$

$y^2=4-4x^2-4z^2$

$y=2\sqrt{1-x^2-z^2}$

$D_2:\begin{cases}0\le x\le 1\\ 0\le z\le\sqrt{1-x^2}\end{cases}$

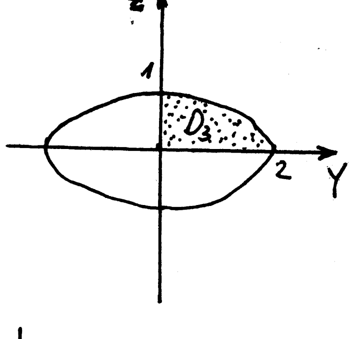
$$J_2=\iint\limits_T y\,dx\,dz=+2\iint\limits_{D_2}\sqrt{1-x^2-z^2}\,dx\,dz=\left|\begin{array}{l}\text{uvedimo polarne}\\ \text{koordinate}\\ x=r\cos\varphi\\ z=r\sin\varphi\\ dz\,dy=r\,dr\,d\varphi\\ D_2\rightsquigarrow D_2'\end{array}\quad D_2':\begin{cases}0\le r\le 1\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}\right|$$

$$=2\iint\limits_{D_2'}\sqrt{1-r^2\cos^2\varphi-r^2\sin^2\varphi}\,r\,dr\,d\varphi=2\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-r^2}\,r\,dr=$$

$$=2\int_0^{\pi/2}d\varphi\int_0^1\sqrt{1-r^2}\left(-\frac{1}{2}\right)d(1-r^2)=-\varphi\Big|_0^{\frac{\pi}{2}}\cdot\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1=-\frac{\pi}{2}\cdot\left(0-\frac{2}{3}\right)=\frac{\pi}{3}$$

$J_3=\iint\limits_T x^2z\,dy\,dz$, vektor normale $\vec{n}$ na površ T sa x-osom zaklapa uglove od 0 do $\frac{\pi}{2}$ pa je $\cos\alpha>0$

Neka je D_3 ortogonalna projekcija površi T na yOz ravan.



$D_3:\ y^2+4z^2=4$ $\quad y^2=4-4z^2$

$\frac{y^2}{4}+\frac{z^2}{1}=1$

$4x^2+y^2+4z^2=4$

$4x^2=4-y^2-4z^2$

$x^2=1-\frac{1}{4}y^2-z^2$

$$J_3=\iint\limits_T x^2z\,dy\,dz=+\iint\limits_{D_3}\left(1-\frac{1}{4}y^2-z^2\right)z\,dy\,dz=$$

$$\left|D_3:\begin{cases}0\le z\le 1\\ 0\le y\le 2\sqrt{1-z^2}\end{cases}\right|=\int_0^1 z\,dz\int_0^{2\sqrt{1-z^2}}\left(1-\frac{1}{4}y^2-z^2\right)dy=\int_0^1 z\left(y\Big|_0^{2\sqrt{1-z^2}}-\frac{1}{4}\cdot\frac{1}{3}y^3\Big|_0^{2\sqrt{1-z^2}}-z^2y\Big|_0^{2\sqrt{1-z^2}}\right)$$

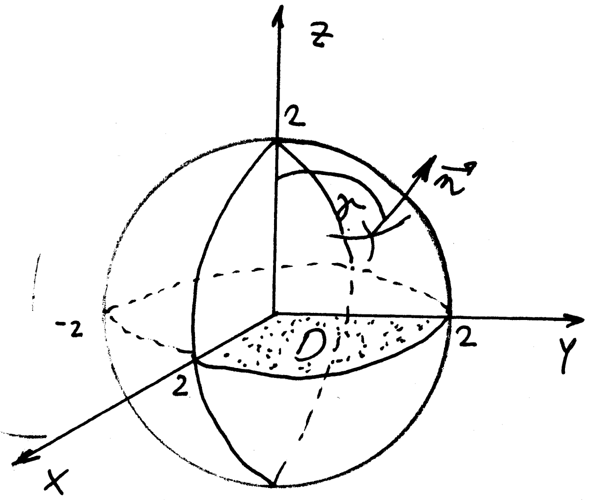
izvučeno $2\sqrt{1-z^2}$

$$=\int_0^1 z\left(2\sqrt{1-z^2}-\frac{2}{3}\sqrt{(1-z^2)^3}-2z^2\sqrt{1-z^2}\right)dz=\frac{4}{3}\int_0^1 z(1-z^2)^{\frac{3}{2}}dz=\frac{2}{3}\cdot\frac{2(1-z^2)^{\frac{5}{2}}}{5}\Big|_1^0=\frac{4}{15}$$

Prema tome $J=\pi+\frac{\pi}{3}-\frac{4}{15}=\frac{4\pi}{3}-\frac{4}{15}$.

Izračunati površinski integral $I=\iint\limits_S x y^3 z\, dx\,dy$, ako je S vanjska strana sfere $x^2+y^2+z^2=4$ u prvom oktantu.

Rj: $x^2+y^2+z^2=4$ je sfera sa centrom u koordinatnom početku čiji je poluprečnik dužine 2.



Kad računamo $\iint\limits_S f(x,y,z)\,dx\,dy$ treba uzeti u obzir predznak broja $\cos\gamma$.
Ako je $\cos\gamma<0$ ispred integrala stavljamo minus, ako je $\cos\gamma>0$ ispred integrala stavljamo plus, a ako je $\cos\gamma=0$ tada je integral jednak 0.
γ je ugao koji vektor normale $\vec{n}$ ($\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$) zaklapa sa z-osom

Vektor normale $\vec{n}$ je u prvom oktantu $\Rightarrow$ $0<\gamma<\frac{\pi}{2}$
$\Rightarrow$ $\cos\gamma>0$

$$x^2+y^2+z^2=4$$
$$z=\pm\sqrt{4-(x^2+y^2)} \quad \text{nama treba } +$$

$$I=\iint\limits_S x y^3 z\, dx\, dy=\iint\limits_D xy^3\left(\sqrt{4-(x^2+y^2)}\right) dx\, dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi \qquad D': \begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 2\end{cases}\\ dx\,dy=r\,dr\,d\varphi\\ \qquad\qquad x^2+y^2=r^2\end{array}\right|$$

$$=\iint\limits_{D'} r\cos\varphi\, r^3\sin^3\varphi\sqrt{4-r^2}\, r\,dr\,d\varphi=\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\, d\varphi\int_0^2 r^5\sqrt{4-r^2}\,dr=I_1\cdot I_2$$

$$I_1=\int_0^{\pi/2}\cos\varphi\cdot\sin^3\varphi\, d\varphi=\left|\begin{array}{l}\sin\varphi=t\\ \cos\varphi\, d\varphi=dt\\ \varphi\big|_0^{\pi/2}\Rightarrow t\big|_0^1\end{array}\right|=\int_0^1 t^3\,dt=\frac{1}{4}t^4\Big|_0^1=\frac{1}{4}$$

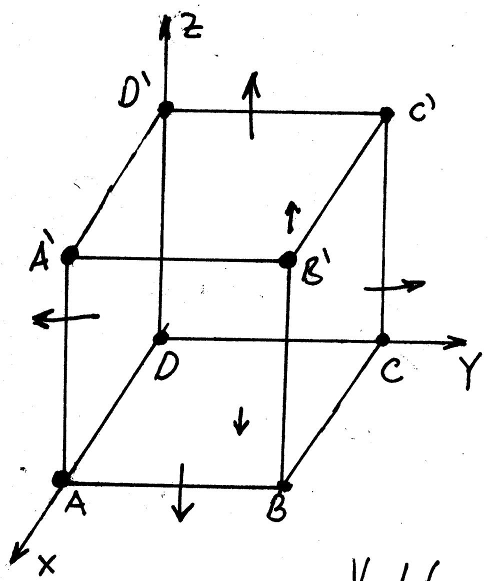
$$I_2=\int_0^2 r^5\sqrt{4-r^2}\,dr=\int_0^2 r^4\sqrt{4-r^2}\,r\,dr=\left|\begin{array}{l}4-r^2=t^2\\ -2r\,dr=2t\,dt\\ r\,dr=-t\,dt\end{array}\quad r\big|_0^2\Rightarrow t\big|_2^0\right|=\int_0^2(4-t^2)^2\cdot t\,t\,dt$$

$$=\int_0^2(16-8t^2+t^4)\cdot t^2\,dt=\int_0^2(t^6-8t^4+16t^2)\,dt=\ldots=\frac{1024}{105}$$

$$I=\frac{1}{4}\cdot\frac{1024}{105}=\frac{256}{105}$$
traženo rješenje

(#) Izračunati integral $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ gdje je S vanjska strana kocke koju čine ravni $x=0$, $y=0$, $z=0$, $x=1$, $y=1$, $z=1$.

Rj.



Označimo sa $I_1 = \iint_S x\,dy\,dz$

Ovaj integral radimo po šest površina: $ABCD$, $ABB'A'$, $BCC'B'$, $ADD'A'$, $A'B'C'D'$ i $DCC'D'$.

Kako imamo $dy\,dz$ posmatramo ugao α koji zaklapa vektor normale na površ sa x osom

Vektor normala površina $ABCD$, $A'B'C'D'$, $BCC'B'$ i $ADD'A'$ je okomita na x-osu $\Rightarrow$

$$\Rightarrow \iint_{ABCD} x\,dy\,dz = \iint_{A'B'C'D'} x\,dy\,dz = \iint_{BCC'B'} x\,dy\,dz = \iint_{ADD'A'} x\,dy\,dz = 0$$

Kako je $x=0$ za površinu $DCC'D'$ $\Rightarrow \iint_{DCC'D'} x\,dy\,dz = 0$

Za I_1 ostaje nam samo površina $ABB'A'$

$$\vec{n}_0 = (1,0,0) \Rightarrow \cos\alpha > 0 \Rightarrow I_1 = +\iint_D dy\,dz$$

gdje je D oblast dobijena projekcijom kvadrata $ABB'A'$ na yOz ravan $D: \begin{cases} 0 \le y \le 1 \\ 0 \le z \le 1 \end{cases}$

$$I_1 = \iint_D dy\,dz = \int_0^1 \left[\int_0^1 dy\right] dz = z\Big|_0^1 \, y\Big|_0^1 = 1$$

Sad nije teško, analognim zaključivanjem, vidjeti da je

$\iint_S y\,dz\,dx = 1$ i $\iint_S z\,dx\,dy = 1$ redom po površinama $BCC'B'$ i $A'B'C'D'$

dok je po ostalim površinama $=0$ $\Rightarrow \iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = 3$

Izračunati površinski integral druge vrste

$$I = \iint_S xyz\,dx\,dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x \geq 0$, $y \geq 0$.

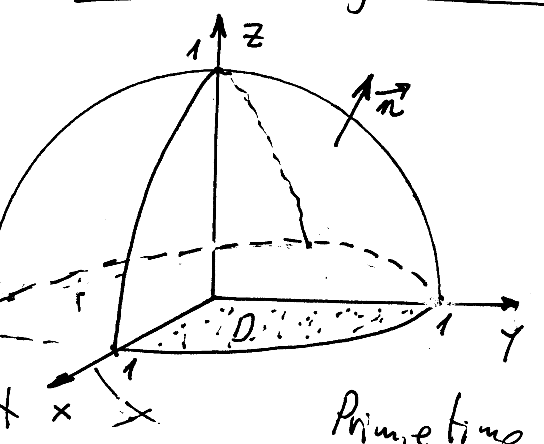
Rj. Prisjetimo se: Neka je površ S data u obliku $z=\eta(x,y)$. Tada

$$\iint_S R(x,y,z)\,dx\,dy = \pm \iint_D R(x,y,\eta(x,y))\,dx\,dy \quad \text{gdje}$$

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0=(\cos\alpha, \cos\beta, \cos\gamma)$,

$\cos\gamma > 0$	$\Rightarrow$	$+$
$\cos\gamma < 0$	$\Rightarrow$	$-$
$\cos\gamma = 0$	$\Rightarrow$	0

- D je ortogonalna projekcija površi S na xOy ravan



$$z^2 = 1-x^2-y^2$$

$$z = \pm\sqrt{1-x^2-y^2}$$

kako je $x \geq 0$, $y \geq 0$ to je $z=\sqrt{1-x^2-y^2}$

Primjetimo da je $0<\gamma<90^\circ \Rightarrow \cos\gamma>0$

$$\iint_S xyz\,dx\,dy = \iint_D xy\sqrt{1-x^2-y^2}\,dx\,dy = \left| \begin{array}{ll} \text{uvedimo polarne koordinate} & \\ x = r\cos\varphi & 1-x^2-y^2 = 1-r^2 \\ y = r\sin\varphi & \\ dx\,dy = r\,dr\,d\varphi & D \xrightarrow{\text{transf.}} D': \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases} \end{array} \right|$$

$$= \iint_{D'} r^3 \sin\varphi\cos\varphi\sqrt{1-r^2}\,dr\,d\varphi = \int_0^1 r^3\sqrt{1-r^2}\,dr \int_0^{\pi/2} \sin\varphi\cos\varphi\,d\varphi \overset{\text{za vježbu}}{=} \ldots = \frac{2}{15}\cdot\frac{1}{2} = \frac{1}{15}$$

Zadaci za vježbu

U zadacima 3887—3893 izračunati date površinske integrale.

3887. $\iint\limits_S x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$ po spoljnoj strani kocke obrazovane ravnima $x=0$, $y=0$, $z=0$, $x=1$, $y=1$. $z=1$.

3888. $\iint\limits_S x^2 y^2 z\,dx\,dy$ po spoljnoj strani donje polovine sfere $x^2+y^2+z^2=R^2$.

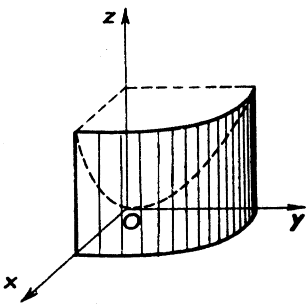
3889. $\iint\limits_S z\,dx\,dy$ po spoljnoj strani elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

3890. $\iint\limits_S z^2\,dx\,dy$ po spoljnoj strani elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

3891. $\iint\limits_S xz\,dx\,dy + xy\,dy\,dz + yz\,dx\,dz$ po spoljnoj strani piramide obrazovane ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$.

3892. $\iint\limits_S yz\,dx\,dy + xz\,dy\,dz + xy\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz dela cilindra $x^2+y^2=R^2$ i odgovarajućih delova ravni $x=0$, $y=0$, $z=0$ i $z=H$.

3893. $\iint\limits_S y^2 z\,dx\,dy + xz\,dy\,dz + x^2 y\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz obrtnog paraboloida $z=x^2+y^2$, cilindra $x^2+y^2=1$ i odgovarajućih delova koordinatnih ravni (sl. 68).



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Rješenja

3887. 3. **3888.** $\frac{2\pi R^7}{105}$. **3889.** $\frac{4}{3}\pi abc$. **3890.** 0.

3891. $\frac{1}{8}$. **3892.** $R^2 H\left(\frac{2R}{3}+\frac{\pi H}{8}\right)$. **3893.** $\frac{\pi}{8}$.